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Monoids over which all flat left acts are regular [1]

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Abstract

Let S be a monoid. It is shown that all flat left S-acts are regular if and only if every cyclic subact of every flat left S-act is strongly flat if and only if every element of S different from 1 is a left zero. This result answers an open question remained in Kilp and Knauer (1987).

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1. Introduction

In the following, S will always stand for a monoid. A left S-act A is called *strongly flat* if the functor $-\otimes A$ (from right S-acts into sets) preserves pullbacks and equalizers. Bulman-Fleming, in [1], showed that preserving only pullback S is sufficient. A left S-act A is called *flat* if the functor $-\otimes A$ preserves embeddings of right S-acts, and *weakly flat* if it preserves embeddings of right ideals into S. According to Knauer [7], a left S-act A is *projective* if and only if $A \cong \coprod Se_i$ for $e_i^2 = e_i \in S$. We note that S in particular is a projective left S-act. As one would expect, projective $\Rightarrow$ strongly flat $\Rightarrow$ flat $\Rightarrow$ weakly flat, and it is also known that these four concepts are distinct.

A left S-act A is called *regular* if for any $a \in A$ there exists a homomorphism $f : Sa \to S$ such that $f(a)a = a$ (cf. [5, 14]). It is clear that a von Neumann regular monoid S is a regular left S-act. Other examples of regular left S-acts were given in [5, 8]. The following result appeared in [5, 14].

Proposition 1.1. *A left S-act A is regular if and only if all cyclic subacts of A are projective.*

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Regular left S-acts were introduced by Tran Lam Hach in [14] and investigated by many authors (for example, see [5, 8, 10, 14]). In [5], it was shown that if S is a *left reversible* monoid (that is, any two principal right ideals of S intersect), then all flat left S-acts are regular if and only if $S = \{1\}$ or $S = \{0,1\}$. But the problem of characterization of monoids over which all flat left S-acts are regular in general remains open (cf. [5]). In this short note, we answer this question by proving that all flat left S-acts are regular if and only if every cyclic subact of every flat left S-act is strongly flat if and only if every element of S different from 1 is left zero. Some results of [5] will be consequences of our investigations.

Any semigroup concept and notation not defined here can be found, for example, in Howie [4].

2. Results

Lemma 2.1 (Bulman-Fleming and Normak [3]). *Every flat cyclic left S-act is strongly flat if and only if S is left nil (that is, for every $x \in S$, $x \neq 1$, there exists a natural number n such that x^n is a left zero element of S).*

Lemma 2.2 (Bulman-Fleming and McDowell [2]). *Let A be a right S-act and B a left S-act. Assume that $a, a' \in A$, $b, b' \in B$. Then $a \otimes b = a' \otimes b'$ in $A \otimes B$ if and only if there exist $a_1, \dots, a_n \in A, b_2, \dots, b_n \in B$, and $s_1, t_1, \dots, s_n, t_n \in S$ such that*

$$\begin{array}{ll} a = a_1 s_1, & \\ a_1 t_1 = a_2 s_2, & s_1 b = t_1 b_2, \\ a_2 t_2 = a_3 s_3, & s_2 b_2 = t_2 b_3, \\ \cdots\cdots & \cdots\cdots \\ a_n t_n = a', & s_n b_n = t_n b'. \end{array}$$

Lemma 2.3. *Let S be a left zero semigroup with 1 adjoined and A a weakly flat left S-act. Suppose that $a \in A$ and $s, t \in S - \{1\}$ are such that $sa = ta$. Then $s = t$.*

Proof. From $sa = ta$, it follows that $s \otimes a = t \otimes a$ in $S \otimes A$. Thus we have $s \otimes a = t \otimes a$ in $(sS \cup tS) \otimes A$ since A is weakly flat. Therefore, by Lemma 2.2, there exist $u_1, v_1, \dots, u_n, v_n \in S, s_1, \dots, s_n \in (sS \cup tS), a_2, \dots, a_n \in A$ such that

$$\begin{array}{ll} s = s_1 u_1, & \\ s_1 v_1 = s_2 u_2, & u_1 a = v_1 a_2, \\ s_2 v_2 = s_3 u_3, & u_2 a_2 = v_2 a_3, \\ \cdots\cdots & \cdots\cdots \\ s_n v_n = t, & u_n a_n = v_n a. \end{array}$$

Set $s_0 = s, s_{n+1} = t, s_i = w_i t_i, w_i \in \{s,t\}, t_i \in S, i = 1, \dots, n$. It is easy to see that there exists an i such that $s_i = st_i, s_{i+1} = tt_{i+1}$. Thus we have

$$s = st_i v_i = s_i v_i = s_{i+1} u_{i+1} = tt_{i+1} u_{i+1} = t,$$

and the result follows.

According to Knauer [6], an element $u \in S$ is called *right semi-cancellative* if for $su = tu, s, t \in S$ there exists $r \in S$ such that $ru = u$ and $sr = tr$.

Lemma 2.4 (Liu and Yang [11, Lemma 3.3]). *Every principal left ideal of S is strongly flat if and only if every element of S is right semi-cancellative.*

The characterization of monoids over which all regular left S-acts are flat was given in [10]. Here we give the characterization of monoids over which all flat left S-acts are regular.

Theorem 2.5. *The following conditions on a monoid S are equivalent:*

(1) *All flat left S-acts are regular.*
(2) *All weakly flat left S-acts are regular.*
(3) *Every cyclic subact of every flat left S-act is strongly flat.*
(4) *Every cyclic subact of every weakly flat left S-act is strongly flat.*
(5) *Every element of S different from 1 is a left zero.*

Proof. The implications (2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (3) are clear. (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) follow from Proposition 1.1.

(5) $\Rightarrow$ (2). If $S = \{1\}$, then the result is clear. Now let S be a left zero semigroup with 1 adjoined. Suppose that A is a weakly flat left S-act and $a \in A$. By Proposition 1.1, our aim will be to show that Sa is a projective left S-act.

Suppose that $sa \neq a$ for any $s \in S - \{1\}$. Define a mapping $f : Sa \to S$ as follows:

$$f(ta) = t, \quad t \in S.$$

Suppose $ta = t'a$. If $t, t' \in S - \{1\}$, then $t = t'$ by Lemma 2.3. If $t \in S - \{1\}$ and $t' = 1$, then $ta = a$, a contradiction. If $t' \in S - \{1\}$ and $t = 1$, the result is similar. This means that f is well-defined. It is clear that f is an isomorphism of left S-acts. Thus Sa is projective.

Now suppose that there exists an element $s \in S - \{1\}$ such that $sa = a$. Define a mapping $f : Sa \to Ss$ as following:

$$f(a) = s,$$

$$f(ta) = t, \quad t \in S - \{1\}.$$

By Lemma 2.3, it is easy to see that f is well-defined. Clearly $f : Sa \to Ss$ is an isomorphism of left S-acts. Thus, by [7], Sa is projective since s is an idempotent of S.

(3) $\Rightarrow$ (5). Since every cyclic subact of every flat left S-act is strongly flat, it is clear that every flat cyclic left S-act is strongly flat. Thus, by Lemma 2.1, S is left nil. Clearly, every principal left ideal of S is strongly flat since S is a flat left S-act. Thus, by Lemma 2.4, every element of S is right semi-cancellative. Suppose that $S \neq \{1\}$

and $x \in S - \{1\}$. Assume n is the least positive integer such that x^n is a left zero element of S. If $n = 1$, then x is a left zero element. Now suppose that $n > 1$. Since

$$x^n x = x^n = x^{n-1}x,$$

there exists $u \in S$ such that

$$x^n u = x^{n-1}u, \qquad ux = x.$$

If $u = 1$, then $x^{n-1} = x^n$, a contradiction. Thus $u \neq 1$. Since S is left nil, there exists a natural number m such that u^m is a left zero element of S. Now we have

$$x = ux = u^2x = \cdots = u^m x = u^m.$$

Thus x is a left zero element of S. This contradicts the hypothesis $n > 1$. Therefore we have proved that S is a left zero semigroup with 1 adjoined.

Remark 2.6. Observing the proof of Theorem 2.5, we obtain that the result is also ture if we replace "all flat" and "all weakly flat" by "all cyclic flat" and "all cyclic weakly flat", respectively, in Theorem 2.5.

Corollary 2.7 (Kilp and Knauer [5, Theorem 2.6]). *Let S be a left reversible monoid. Then all flat left S-acts are regular if and only if $S = \{1\}$ or $S = \{0, 1\}$.*

A monoid S is called a *right PP monoid* if every principal right ideal of S is projective. It was shown in [3] that if S is a right PP monoid such that all cyclic flat left S-acts are strongly flat, then every element of S different from 1 is a left zero. From Theorem 2.5, it follows that the converse is also ture. That is, we have:

Corollary 2.8. *The following conditions on S are equivalent:*
(1) *S is a right PP monoid such that all cyclic flat left S-acts are strongly flat.*
(2) *Every element of S different from* 1 *is a left zero.*

Remark 2.9. It was shown in [10] that all regular left S-acts are flat if and only if se is a von Neumann regular element of S for all $s \in S$ and $e^2 = e \in T$, where T is the largest regular left ideal of S. Thus, from Theorem 2.5, it follows that if all flat left S-acts are regular, then all regular left S-acts are flat. The following example shows that there exists a monoid S such that all regular left S-acts are flat but not all flat left S-acts are regular. Set $S = P \dot{\cup} T$, where P is a monoid with only one idempotent which is not right cancellative, T a von Neumann regular semigroup and $pt = tp = t$ for all $p \in P$, $t \in T$. It was shown in [10] that all regular left S-acts are flat. But, there exists a flat S-act which is not regular.

Let A be a left S-act. Normak, in [12], studied the following condition (E), which was first considered by Stenstrom [13].

(E) If $sa = ta$ for $a \in A, s, t \in S$ then there exist $a' \in A$ and $u \in S$ such that $su = tu$ and $a = ua'$.

In [9], it was proved that all left S-acts satisfying condition (E) are flat if and only if S is von Neumann regular. But the problem of characterization of monoids over which all flat left S-acts satisfy condition (E) remains open (see [9, Remark 5]). Here, by Theorem 2.5, we have a partial answer for this problem.

Corollary 2.10. *Suppose that every principal left ideal of S is strongly flat. Then the following conditions are equivalent:*
(1) *All flat left S-acts satisfy condition (E).*
(2) *All weakly flat left S-acts satisfy condition (E).*
(3) *Every element of S different from* 1 *is a left zero.*

Proof. (1) $\Rightarrow$ (3). It was proved in [6] that a cyclic left S-act Sx is strongly flat if and only if for $sx = tx, s, t \in S$ there exists $r \in S$ with $rx = x$ and $sr = tr$. Thus it is easy to see that a cyclic left S-act is strongly flat if and only if it satisfies condition (E). Therefore every flat cyclic left S-act is strongly flat. Now the implication follows from the proof of Theorem 2.5, part (3) $\Rightarrow$ (5).

(3) $\Rightarrow$ (2). By Theorem 2.5, every cyclic subact of every weakly flat left S-acts is strongly flat. Thus the result follows.

(2) $\Longrightarrow$ (1). Clear.

References

[1] S. Bulman-Fleming, Pullback-flat acts are strongly flat, Canad. Math. Bull. 34 (1991) 1–6.

[2] S. Bulman-Fleming and K. McDowell, Absolutely flat semigroups, Pacific J. Math. 107 (1983) 319–333.

[3] S. Bulman-Fleming and P. Normak, Monoids over which all flat cyclic right acts are strongly flat, Semigroup Forum 50 (1995) 233–241.

[4] J.M. Howie, An Introduction to Semigroup Theory (Academic Press, New York, 1976).

[5] M. Kilp and U. Knauer, Characterization of monoids by properties of regular acts, J. Pure Appl. Algebra 46 (1987) 217–231.

[6] U. Knauer, Characterization of monoids by properties of finitely generated right acts and their right ideals, Lecture Notes in Math., Vol. 998 (Springer, Berlin, 1983) 310–332.

[7] U. Knauer, Projectivity of acts and Morita equivalence of monoids, Semigroup Forum 3 (1972) 359–370.

[8] U. Knauer and A. Mikhalev, Wreath products of acts over monoids, I. Regular and inverse acts, J. Pure Appl. Algebra 51 (1988) 251–260.

[9] Z.K. Liu, A characterization of regular monoids by flatness of left acts, Semigroup Forum 46 (1993) 85–89.

[10] Z.K. Liu, Monoids over which all regular left acts are flat, Semigroup Forum 50 (1995) 135–139.

[11] Z.K. Liu and Y.B. Yang, Monoids over which every flat right act satisfies condition (P), Comm. Algebra 22(8) (1994) 2861–2875.

[12] P. Normak, On equalizer-flat and pullback-flat acts, Semigroup Forum 36 (1987) 293–313.

[13] Bo Stenstrom, Flatness and localization over monoids, Math. Nachr. 48 (1972) 315–334.

[14] Tran Lam Hach, Characterization of monoids by regular acts, Period. Sci. Math. Hung. 16 (1985) 273–279.